Solution Sheet 5

Exercise 5.1

Let T be a linear operator on a separable Hilbert Space H such that for some orthonormal basis (e_i) of H,

$$\sum_{i=1}^{\infty} \langle Te_i, e_i \rangle < \infty.$$

Is T necessarily compact?

Proof. No; define T to be the shift operator, $Te_i := Te_{i+1}$. Then

$$\sum_{i=1}^{\infty} \langle Te_i, e_i \rangle = \sum_{i=1}^{\infty} \langle e_{i+1}, e_i \rangle = 0.$$

However, T is not compact: the sequence (e_i) is bounded in H, though $(Te_i) = (e_{i+1})$ has no convergent subsequence. □

Exercise 5.2

1. Suppose that Y is normally distributed with mean zero and variance one. Is the process X defined by $X_t = \sqrt{t}Z$ a Brownian Motion?
2. Let B, W denote two independent standard real-valued Brownian Motions. For $\lambda \in [0, 1]$ define the process Z by
$$Z_t = \lambda B_t + \sqrt{1 - \lambda^2} W_t.$$
Is Z a Brownian Motion?

Proof.

1. No; for $s < t$, $X_t - X_s = (\sqrt{t} - \sqrt{s})Y \sim N\left(0, (\sqrt{t} - \sqrt{s})^2\right) \neq N(0, t - s)$.
2. Yes; Z is continuous and starting from zero as the sum of two such processes. For independence of increments, we use that increments of B are independent from one another and that as B is independent of W then all increments of B are further independent from all increments of W, hence all finite linear combinations preserve independence. For the distribution of increments of Z, we observe that
$$Z_t - Z_s = \lambda(B_t - B_s) + \sqrt{1 - \lambda^2}(W_t - W_s)$$
which is the sum of independent normally distributed random variables, $N(0, \lambda^2(t - s))$ and $N(0, (1 - \lambda^2)(t - s))$ respectively, hence the sum has distribution $N(0, t - s)$.

□

Exercise 5.3

For any fixed $T > 0$, and for W a standard real-valued Brownian Motion, define the process B on $[0, T]$ by

$$B_t = W_t - \frac{t}{T} W_T.$$

1. Prove that B is a Gaussian process on $[0, T]$.
2. Calculate the covariance of B.
3. Show that B is independent of the process $\tilde{W}$ defined by $\tilde{W}_t = \tilde{W}_{t+T}$.
4. Demonstrate that the process $\tilde{B}$ defined on $[0, T)$ by

$$\tilde{B}_t = \frac{T-t}{\sqrt{T}} W_{\frac{t}{T-t}}$$

has the same law as B.

5. Let $(\mathcal{F}_t^B)$ denote the completed filtration generated by B. Verify that for $0 \leq s < t \leq T$,

$$\mathbb{E}(B_t | \mathcal{F}_s^B) = \frac{T-t}{T-s} B_s.$$

6. Define the process V on $[0, T)$ by

$$V_t = \int_0^t \frac{B_r}{T-r} dr.$$

Confirm that the process X on $[0, T)$ defined by

$$X_t = B_t + V_t$$

is a standard Brownian Motion for $(\mathcal{F}_t^B)$.

Proof.

1. For any finite collection of times $0 \leq t_1 < \cdots < t_n \leq T$, the vector $(B_{t_1}, \ldots, B_{t_n})$ can be obtained from finite linear combinations of vectors from the Gaussian process W.
2. Via direct computation,

$$\begin{aligned}
\mathbb{E}(B_s B_t) &= \mathbb{E}\left[\left(W_s - \frac{s}{T} W_T\right)\left(W_t - \frac{t}{T} W_T\right)\right] \\
&= \mathbb{E}(W_s W_t) - \frac{t}{T}\mathbb{E}(W_s W_T) - \frac{s}{T}\mathbb{E}(W_T W_t) + \frac{st}{T^2}\mathbb{E}(W_T^2) \\
&= (s \wedge t) - \frac{t}{T} s - \frac{s}{T} t + \frac{st}{T^2} T \\
&= (s \wedge t) - \frac{st}{T}.
\end{aligned}$$

3. As argued in the first part, B and W are jointly Gaussian. Therefore it is sufficient to show that they have null covariance:

$$\mathbb{E}(B_s \tilde{W}_t) = \mathbb{E}(W_s W_{t+T}) - \frac{s}{T}\mathbb{E}(W_T W_{t+T}) = s - \frac{s}{T} T = 0.$$

4. $\tilde{B}$ is a centred Gaussian process, inherited from W, so its law is characterised by the covariance

$$\begin{aligned}\mathbb{E}(\tilde{B}_s\tilde{B}_t) &= \mathbb{E}\left[\frac{T-s}{\sqrt{T}}W_{\frac{s}{T-s}}\frac{T-t}{\sqrt{T}}W_{\frac{t}{T-t}}\right]\\ &= \frac{(T-t)(T-s)}{T}\left[\left(\frac{s}{T-s}\right)\wedge\left(\frac{t}{T-t}\right)\right]\\ &= \frac{(T-t)(T-s)}{T}\frac{(s\wedge t)}{T-(s\wedge t)}.\end{aligned}$$

For the sake of clarity let us assume that $(s \wedge t) = s$. Then the above is

$$\frac{(T-t)(T-s)}{T}\frac{s}{T-s} = \frac{(T-t)s}{T} = s - \frac{st}{T}$$

which matches the covariance of B shown in part 2, as required.

5. We look to show that

$$\mathbb{E}\left(B_t - \frac{T-t}{T-s}B_s\middle|\mathcal{F}_s^B\right) = 0. \tag{1}$$

Observe that

$$B_t - \frac{T-t}{T-s}B_s = W_t - \frac{t}{T}W_T - \frac{T-t}{T-s}\left[W_s - \frac{s}{T}W_T\right] = W_t - \frac{T-t}{T-s}W_s + \frac{s-t}{T-s}W_T. \tag{2}$$

Let $(\mathcal{F}_t^W)$ denote the completed filtration of W. From the definition of B clearly $\mathcal{F}_s^B \subset \mathcal{F}_s^W \vee \sigma(W_T)$, so we use the tower property of conditional expectation to see that

$$\mathbb{E}\left(B_t - \frac{T-t}{T-s}B_s\middle|\mathcal{F}_s^B\right) = \mathbb{E}\left[\mathbb{E}\left(B_t - \frac{T-t}{T-s}B_s\middle|\mathcal{F}_s^W \vee \sigma(W_T)\right)\middle|\mathcal{F}_s^B\right].$$

From the independence shown in part 3, $B_t - \frac{T-t}{T-s}B_s$ is independent of $\sigma(W_T)$ hence

$$\mathbb{E}\left(B_t - \frac{T-t}{T-s}B_s\middle|\mathcal{F}_s^W \vee \sigma(W_T)\right) = \mathbb{E}\left(B_t - \frac{T-t}{T-s}B_s\middle|\mathcal{F}_s^W\right).$$

Using the representation (2),

$$\begin{aligned}\mathbb{E}\left(B_t - \frac{T-t}{T-s}B_s\middle|\mathcal{F}_s^W\right) &= \mathbb{E}\left(W_t - \frac{T-t}{T-s}W_s + \frac{s-t}{T-s}W_T\middle|\mathcal{F}_s^W\right)\\ &= W_s - \frac{T-t}{T-s}W_s + \frac{s-t}{T-s}W_s\\ &= 0\end{aligned}$$

confirming (1).

6. Note that V is well defined as the integrand is bounded by pathwise continuity of B. We look to use Lévy's Characterisation of Brownian Motion; firstly then we must show that X

is a martingale. Indeed for $0 \leq s < t < T$, by Fubini's Theorem and the previous part,

$$\begin{aligned}
\mathbb{E}\left(V_t - V_s | \mathcal{F}_s^B\right) &= \mathbb{E}\left(\int_s^t \frac{B_r}{T-r} dr \Big| \mathcal{F}_s^B\right) \\
&= \int_s^t \frac{\mathbb{E}\left(B_r | \mathcal{F}_s^B\right)}{T-r} dr \\
&= \int_s^t \frac{1}{T-r} \frac{T-r}{T-s} B_s dr \\
&= \int_s^t \frac{B_s}{T-s} dr \\
&= \frac{t-s}{T-s} B_s.
\end{aligned}$$

Using the previous part once more,

$$\mathbb{E}\left(B_s - B_t | \mathcal{F}_s^B\right) = \left(1 - \frac{T-t}{T-s}\right) B_s = \frac{t-s}{T-s} B_s = \mathbb{E}\left(V_t - V_s | \mathcal{F}_s^B\right)$$

therefore $\mathbb{E}(X_t - X_s | \mathcal{F}_s^B) = 0$ so X is an $(\mathcal{F}_t^B)$ martingale. It only remains to show that the quadratic variation $[X]_t = t$, for which we note that V is of finite variation hence $[X]_t = [B]_t$, but of course $\left(\frac{t}{T} W_T\right)$ is also of finite variation so $[B]_t = [W]_t = t$ as required.

$\square$

Exercise 5.4

Verify that each λ_k, e_k defined by

$$\lambda_k = \left(\left(k - \frac{1}{2}\right)\pi\right)^{-2}, \qquad e_k(t) = \sqrt{2}\sin\left(\left(k - \frac{1}{2}\right)\pi t\right)$$

are respective eigenvalues and eigenfunctions of the covariance operator for Brownian Motion on $[0,1]$.

Proof. We need to verify that

$$\int_0^1 (s \wedge t) e_k(t) dt = \lambda_k e_k(s)$$

which we rewrite as

$$\int_0^s t e_k(t) dt = \int_s^1 e_k(t) dt = \lambda_k e_k(s)$$

from which we directly plug in the values of λ_k and e_k to yield the result.

$\square$

Exercise 5.5

Compute

$$\mathbb{E}\left(e^{-\frac{1}{2}\int_0^1 W_t^2 dt}\right).$$

Proof. We use the expansion of W specified in Proposition 2.6.11, for (λ_k, e_k) as in Exercise 5.4, that is

$$W_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k e_k(t)$$

where (β_k) are i.i.d standard Gaussian random variables. Then

$$\begin{aligned}
\int_0^1 W_t^2 dt &= \int_0^1 \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \sqrt{\lambda_k \lambda_l} \beta_k \beta_l e_k(t) e_l(t) dt \\
&= \sum_{k=1}^{\infty} \int_0^1 \lambda_k \beta_k^2 e_k^2(t) dt \\
&= \sum_{k=1}^{\infty} \lambda_k \beta_k^2
\end{aligned}$$

using that the (e_k) form an orthonormal basis of $L^2([0,1];\mathbb{R})$. Therefore

$$\mathbb{E}\left(e^{-\frac{1}{2}\int_0^1 W_t^2 dt}\right) = \mathbb{E}\left(\Pi_{k=1}^{\infty} e^{-\frac{1}{2}\lambda_k \beta_k^2}\right) = \Pi_{k=1}^{\infty} \mathbb{E}\left(e^{-\frac{1}{2}\lambda_k \beta_k^2}\right)$$

using that the (β_k) are independent. As they are standard normal, then

$$\mathbb{E}\left(e^{-\frac{1}{2}\lambda_k \beta_k^2}\right) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot e^{-\frac{1}{2}\lambda_k x^2} dx = \sqrt{\frac{1}{1+\lambda_k}}$$

hence

$$\mathbb{E}\left(e^{-\frac{1}{2}\int_0^1 W_t^2 dt}\right) = \Pi_{k=1}^{\infty} \sqrt{\frac{1}{1+\lambda_k}}.$$

One can simplify this further with the explicit form of λ_k to $\sqrt{\frac{1}{\cosh(1)}}$.

□